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Horizon-switching Predictive Set-point Tracking under Input-increment Saturations and Persistent Disturbances

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Predictive switching logic schemes are considered whereby a feedback-gain is switched-on at any time from a family of candidate feedback-gains so as to deal with the problem of set-point tracking in Asymptotically Null-controllable with Bounded Input (ANCBI) systems under input-increment saturations and persistent arbitrary bounded disturbances. It is constructively shown that such schemes do exist which ensure, along with good tracking performance, global asymptotic and semi-global exponential stability in the noiseless case, as well as bounded-noise bounded-state l_∞ -stability, whenever the disturbances enter the system in such a way to make these properties conceptually achievable.

Keywords: Anti-windup control, control-saturated systems, nonlinear control, switching control, predictive control.

1. Introduction

Control of input-saturated dynamic systems, though a fundamental issue in control engineering, has been given exhaustive and constructive systematic answers mainly only during the last 15 years.

From one side it was characterized the class of dynamical linear time-invariant (LTI) systems whose state can be asymptotically driven to zero with arbitrarily small controls [12,26]: the so called ANCBI

(Asymptotically Null-controllable with Bounded Input) systems. In discrete-time, they coincide with all stabilizable LTI systems with eigenvalues on the closed unit disk. Hence, they encompass stable systems with integrator chains of arbitrary complexity, and are representative of a great deal of processes of practical interest [27]. From another side, linear control structures were shown to only provide semi-global stabilization of input-saturated ANCBI systems [17,20]. Non-linear state feedback schemes for input-saturated ANCBI plants were discussed in [25,26]. However, such schemes amount to low-gain control strategies which feature poor regulation performance. In an attempt to provide enhanced performance, gain-scheduling variants, akin to the approach adopted in [21] and the present paper, were proposed in [1,18]. The main interest of a different line of research has been on anti-windup control schemes [7,8,9,28] whereby suitable corrections to a pre-designed compensator are generated whenever input saturations take place.

While positional input saturations have attracted a great deal of interest, fewer results apply to incremental input saturations. Input-increment saturations are a serious challenge in many automatic control applications, *e.g.*, [19], joint position and rate saturated control [20,29], flight control [6,15]. In particular, it is known [3] that they can induce a significant destabilizing effects due to phase-lag.

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However, it is to be pointed out that all these contributions mainly deal with the stabilization issue, and put little emphasis on the performance of the overall controlled system. Moreover, apart from a few exceptions [2,4,24], little attention has been devoted on how to deal with persistent disturbances of unknown arbitrary magnitude. In such a context switching control appears to be important not only for enhancing performance but also with respect to the stabilization issue [10].

A recent paper [21], though, restricted to the pure regulation case, has reconsidered the problem from the viewpoint of both stability and performance of systems subject to positional input saturations and affected by persistent disturbances of arbitrary unknown intensity. The algorithm there proposed enjoys the following features:

1. It is realized via a supervisory switching control scheme whereby a feedback-gain, selected from a finite family of pre-designed candidate feedback-gains, is at any time switched-on in feedback to the plant according to the previous feedback-gain and the information, either complete or partial, on the current plant state;
2. No disturbance upper-bound need to be known;
3. The feedback-gain selection is made in accordance with a predictive control philosophy, and each candidate feedback-gain is tuned on to a different control-horizon;
4. The supervisory switching logic is flexible enough so as to enable the designer to simplify the scheme by trading off performance vs. memory and/or computational complexity, while retaining guaranteed stability properties.

The present paper aims at adapting the approach of [21] to the case of set-point tracking for LTI systems subject to incremental input saturations and also affected by persistent disturbances of unknown arbitrary magnitude. The paper is organized as follows. Section 2 describes the so-called incremental model of the system to be controlled and formulates the problem. Section 3 introduces the specific type of feedback-gain matrices that are adopted to realize possible control actions, and describes some property of the control law. Section 4 shows that the algorithm proposed in [21] can be used also in the presence of both input-increment saturation and persistent disturbances for the set-point tracking problem. It also remarks how to deal with partial state information and how to keep the memory requirement and the computational load at a moderate level. Section 5 reports a simulation example which illustrates the

effectiveness of the technique proposed. Section 6 ends the paper with conclusive remarks.

2. Preliminaries and Problem Formulation

The following notations will be used throughout the paper: $\overleftarrow{m} := \{1, 2, \dots, m\}$, with $m \geq 1$; $\|x\| = [x'x]^{1/2}$ is the usual Euclidean norm, where the prime denotes transpose; $\lambda_i(\Phi)$, $i \in \overleftarrow{n}$, $n = \dim(\Phi)$, and $\text{sp}(\Phi)$ denote the i^{th} Φ -eigenvalue and, respectively, the whole set of the eigenvalues of Φ .

Consider the following discrete-time LTI *Underlying Positional System* (UPS)

$$\begin{cases} s(t+1) = \Phi_s s(t) + G_s u(t) + \xi(t) \\ q(t+1) = \Phi_q q(t) + G_q u(t) + c(t). \end{cases} \quad (1)$$

where $\Phi_s \in \mathbb{R}^{n_s}$ and $\Phi_q \in \mathbb{R}^{n_q}$; $x(t) := [s'(t)q'(t)]'$, $x(t) \in \mathbb{R}^{n_x}$, $n_x := n_s + n_q$ is the plant state; $|\lambda_i(\Phi_s)| < 1$, $i \in \overleftarrow{n_s}$, $|\lambda_j(\Phi_q)| = 1$, $j \in \overleftarrow{n_q}$, with arbitrary geometric multiplicities; $u(t) \in \mathbb{R}^m$ is the control. Let

$$e(t) := y(t) - r(t) \quad (2)$$

denote the tracking error, $y(t) = H_s s(t) + H_q q(t) + \zeta(t)$, $y(t) \in \mathbb{R}^m$ being the performance variable (output), and $r(t)$ the set-point to be tracked by the output. The vectors $\xi(t)$, $c(t)$ and $\zeta(t)$ represent arbitrary bounded disturbances. The following assumption is adopted:

$$(\Phi, G) \text{ reachable} \quad (3)$$

where $\Phi := \text{Diag}\{\Phi_s, \Phi_q\}$, $G := [G'_s G'_q]'$. Notice that, in the case of a pure regulation problem, systems of the form (1), provided that $c(t) \equiv 0$, have the most general structure in order to achieve state boundedness under the condition of saturated control and boundedness of arbitrary disturbances (e.g., [11,23]).

2.1. Problem Formulation

The paper aims at finding a feedback control action which ensures global asymptotic stability and offset-free tracking in the presence of constant disturbances and set-point r , as well as finite l_∞ induced gain to the disturbance-to-state map in the presence of time-varying disturbances and set-point, while preserving the fulfillment of control-increment saturation constraints.

According to the internal model principle, a classic approach adopted in automatic control to track a constant reference r is to enforce an “integral action” in the feedback loop. The related design hinges upon the so-called *incremental model* (IM_e)

$$\begin{cases} \chi(t+1) = A\chi(t) + B\delta u(t) + \delta v(t) \\ e(t) = C\chi(t) + \delta w(t) \end{cases} \quad (4)$$

with $t \in \mathbb{Z}_{-1} := \{-1, 0, 1, \dots\}$

$$A := \begin{bmatrix} \Phi & 0 \\ H & I_m \end{bmatrix} \quad B := \begin{bmatrix} G \\ 0_m \end{bmatrix} \quad C := [H \quad I_m] \quad (5)$$

$\chi(t) := [\delta x'(t)e'(t-1)]' \in \mathbb{R}^n$, where $n := n_x + m$ and $\delta x(t) := x(t) - x(t-1)$; exogenous signals $\delta v(t) := [\delta \xi'(t) \delta c'(t) \delta w'(t)]'$, $\delta w(t) := \delta \zeta(t) - \delta r(t)$. Finally, the control signal $\delta u(t) \in \mathbb{R}^m$ is supposed to be subject to the following saturation constraint

$$\delta u \in D := \{\delta u \in \mathbb{R}^m : \Delta^- < \delta u < \Delta^+\} \quad (6)$$

where $\Delta^- := [\Delta_1^-, \dots, \Delta_m^-]'$ and $\Delta^+ := [\Delta_1^+, \dots, \Delta_m^+]'$ with $\Delta_i^-, \Delta_i^+ \geq 0$, $i \in \bar{m}$. Notice that the vector inequality in (6) is to be interpreted in a component-wise sense. By the sake of simplicity, hereafter it will be assumed that $\bar{\Delta} = \Delta^+ = -\Delta^- < \infty$. All the results that follow can be extended to the non-symmetric case via suitable technicalities.

It is well known that a linear state-feedback law $\delta u(t) = \mathcal{F}\chi(t)$, which stabilizes (4), ensures an offset-free steady-state tracking error for the class of constant disturbances and references. As for (4), direct application of a PBH rank test (see [13]) shows that (A, B) is reachable if and only if (Φ, G) is such and

$$\det \begin{bmatrix} I_{n_x} - \Phi & G \\ H & 0_m \end{bmatrix} \neq 0 \quad (7)$$

The latter is a necessary and sufficient condition for the existence of a stabilizing linear state-feedback with integral action for system (1) under (3) (see [4,5]).

2.2. Conceptually Achievable Results

In order to possibly achieve the stated goals it is to be pointed out that system (4) has not seemingly the structure of an input-saturated LTI system for which it makes sense to consider stability and boundedness under arbitrary l_∞ -disturbances. Hence, it is not possible to be sure that the direct adoption to the present case of the approach [21] can achieve such goals. Indeed, suppose temporarily that $\delta w(t) \equiv 0$, $\forall t \in \mathbb{Z}_+$, with $\delta \xi(t)$ entering only the stable modes of

$\chi(t)$. Even if this was the case, the neutrally stable modes of $\chi(t)$ would be indirectly affected by $\delta \xi(t)$ via the stable ones. However, one has to take account that (4) is a representation for design of the real UPS (1). Hence, not only the positional and the incremental instantaneous values of the disturbances are bounded but also their own incremental sums. As will be seen this is the property that allows one to prove the conjecture that the algorithm in [21] can still work in the present case of set-point tracking.

Nevertheless, it is worth pointing out that the present problem of set-point tracking is subject to the same intrinsic limitation, similarly to the pure regulation problem. Indeed, as can be checked, the time-varying disturbance $\delta c(t)$, entering the neutrally stable modes of $\chi(t)$ in (4), is generally not allowed to assume arbitrary values, e.g., $c(t)$ cannot assume arbitrary incremental values¹. Consequently, in the following it will be assumed that such a disturbance $c(t)$ in (1) be constant, i.e., $c(t) \equiv c$. Only a pertinent result of [21] will be mentioned for the case of time-varying $c(t)$.

3. Receding Horizon Feedback Gains

In connection with the incremental model (4), let χ be its state at time 0, and $\Omega_h(\chi)$ the set of all control increments ω of length h , $\omega = [\delta u'(0), \dots, \delta u'(h-1)]'$, which drive the system state to the zero-state 0_χ in h time-steps

$$\Omega_h(\chi) := \left\{ \omega \in (\mathbb{R}^m)^h : \chi(h) = 0_\chi \right\} \quad (8)$$

where $\chi(h) = \mathcal{A}^h \chi + \sum_{k=0}^{h-1} \mathcal{A}^{h-1-k} B \delta u(k)$. Note that $\Omega_h(\chi) \neq \emptyset, \forall \chi \in \mathbb{R}^n$, $n := n_x + m$, if $h \geq \nu$, with $\nu, \nu \leq n$, the reachability index of (A, B) . Let $\delta u_h(\chi)$ the element in $\Omega_h(\chi)$ of minimum energy

$$\sum_{k=0}^{h-1} \delta u'(k) \Psi_u \delta u(k) = \omega' \hat{\Psi}_u \omega \quad (9)$$

where $\hat{\Psi}_u := \text{Diag}\{\Psi_u, \dots, \Psi_u\}$ (h -times), $\Psi_u = \Psi_u' > 0$. For $h \geq \nu$, $\delta u_h(\chi)$ is as follows

$$\begin{aligned} \delta u_h(\chi) &:= [\delta u'_h(0|\chi), \dots, \delta u'_h(h-1|\chi)]' \\ &= [\mathcal{F}'_h(0) \cdots \mathcal{F}'_h(h-1)]' \chi \\ &= \mathcal{F}_h \chi \end{aligned} \quad (10)$$

¹Consider the system $x(t+1) = -x(t) + u(t) + c(t)$, $x(0) = x_0$, $c(t) = \bar{c} \cdot (-1)^t$, $\bar{c} > 0$, with $u(t)$ subject to (6). If $\bar{\Delta} < \bar{\Delta}_c := 2\bar{c}$ the regulation problem has no solution.

$$\mathcal{F}_h := -\hat{\Psi}_u^{-1} \mathcal{R}_h' \mathcal{G}_h^{-1} \mathcal{A}^h \quad (11)$$

where \mathcal{R}_h is the h -order reachability matrix

$$\mathcal{R}_h := [\mathcal{A}^{h-1} \mathcal{B} | \dots | \mathcal{A} \mathcal{B} | \mathcal{B}] \quad (12)$$

and \mathcal{G}_h the h -order reachability Gramian

$$\mathcal{G}_h := \mathcal{R}_h \hat{\Psi}_u^{-1} \mathcal{R}_h' \quad (13)$$

The integer h will be referred to as the *control horizon*. Let F_h be as follows

$$F_h = [I_m \quad 0_{m \times m(h-1)}] \mathcal{F}_h = \mathcal{F}_h(0) \quad (14)$$

Hence, F_h is recognized to be the feedback-gain matrix of the receding horizon regulation related to the zero-terminal state minimum energy control problem of horizon h . Notice that $\forall t \in \mathbb{Z}_+$

$$u_{h(t)}(k|\chi(t)) = u(t-1) + \sum_{i=0}^k \delta u_{h(t)}(i|\chi(t)) \quad (15)$$

being $h(t)$ the control horizon at time t . The latter equation yields, at each iteration step, the control law for the system (1)

$$u(t) := u(t-1) + \delta u_{h(t)}(0|\chi(t)) \quad (16)$$

Let

$$M_h(\chi) := \max \left\{ \frac{|\delta u_h(k|\chi)|}{\Delta}; k+1 \in \overleftarrow{h}; i \in \overleftarrow{m} \right\} \quad (17)$$

where $[\delta u]_i$ denotes the i^{th} component of the vector δu . Note that the whole sequence $\delta u_h(\chi)$ does not violate (6) if and only if $M_h(\chi) \leq 1$. If (4) is ANCB1, it is always possible to find a large enough horizon h so as to satisfy $M_h(\chi) \leq 1$. In fact, it can be shown [21] that for an ANCB1 system

$$M_h(\chi) \leq \overline{M} h^{-1} \|\chi\| \quad (18)$$

where \overline{M} is a positive real depending on $(\mathcal{A}, \mathcal{B})$.

Consider the incremental model (4) and a similarity transformation $\mathcal{T} : \chi \rightarrow \chi_\varepsilon := [\delta s' \delta q' \varepsilon']'$ (ε^- stands for $\varepsilon(t-1)$), under which (4) is diagonalized w.r.t. δs

$$\mathcal{T} := \begin{bmatrix} I_s & 0 & 0 \\ 0 & I_q & 0 \\ H_s(I_s - \Phi_s)^{-1} & 0 & I_m \end{bmatrix} \quad (19)$$

Such a transformation always exists as $1 \notin \text{sp}(\Phi_s)$. As can be checked, this leads to the next incremental model (IM $_\varepsilon$) algebraically equivalent to (4)

$$\begin{cases} \delta s(t+1) = \Phi_s \delta s(t) + G_s \delta u(t) + \delta \xi(t) \\ \delta q(t+1) = \Phi_q \delta q(t) + G_q \delta u(t) \\ \varepsilon(t) = \varepsilon(t-1) + H_q \delta q(t) + W_s \delta u(t) + \delta n(t) \end{cases} \quad (20)$$

where $W_s := H_s(I_s - \Phi_s)^{-1} G_s$, $\delta n(t) := \hat{W}_s \delta \xi(t) + \delta \zeta(t) - \delta r(t)$, and $\hat{W}_s := H_s(I_s - \Phi_s)^{-1}$. Similarly to (5),

$$\mathcal{A} := \begin{bmatrix} \Phi & 0 \\ \hat{H} & I_m \end{bmatrix} \quad \mathcal{B} := \begin{bmatrix} G \\ W_s \end{bmatrix} \quad \mathcal{C} := [\hat{H} \quad I_m] \quad (21)$$

with $\hat{H} := [0 \quad H_q]$. Because (10) is linear in χ_ε ,

$$\delta u_h(\chi_\varepsilon) = \delta u_h^s(\delta s) + \delta u_h^q(\delta q) + \delta u_h^\varepsilon(\varepsilon^-) \quad (22)$$

Thus, by (20) one has $\delta u_h^s(k|\delta s) := -\Psi_u^{-1} \mathcal{B}' (\mathcal{A}^{h-k-1})' \times \mathcal{G}_h^{-1} \delta \hat{s}$ where $\delta \hat{s} := [(\Phi_s^h \delta s)' 0_q' 0_\varepsilon']'$, and similarly for $\delta u_h^q(\delta q)$. Moreover $\delta u_h^\varepsilon(k|\varepsilon^-) := -\Psi_u^{-1} \mathcal{B}' (\mathcal{A}^{h-k-1})' \times \mathcal{G}_h^{-1} \hat{\mathcal{A}}^h \chi_\varepsilon$, where $\hat{\mathcal{A}}^h := \begin{bmatrix} 0_s & 0 & 0 \\ 0 & 0_q & 0 \\ 0 & H_q \sum_{i=0}^{h-1} \Phi_q^i & I_m \end{bmatrix}$. It is

to be pointed out that (22) allows one to consider separately the contribution in $\delta u_h(\chi_\varepsilon)$ given by the disturbances $\delta \xi$ injected in the stable modes and the contribution caused by the disturbances δn affecting the critically unstable ones.

Remark 1. It is to be pointed out that the main results of this paper, i.e., Theorem 1 and Theorem 2, apply to the real UPS (1). Both models (4) and (20) are representations for design and analysis of (1).

4. Hysteresis Switching Logic

The question to be posed is whether the approach of [21], restricted to the pure regulation problem, can also work in the present context of set-point tracking problem. Let $\delta u(t) = F_{h(t)} \chi_\varepsilon(t)$ with F_h as in (14) and $h(t)$ chosen according to the following *hysteresis switching logic* ($\underline{h} \geq n, n = n_x + m$)

$$h(t) = \begin{cases} \tilde{h}(t), & \text{if } M_{\tilde{h}(t)}(\chi_\varepsilon(t)) \leq 1 \\ \hat{h}(t), & \text{otherwise.} \end{cases} \quad (23)$$

$$\tilde{h}(t) := \max\{\underline{h}, h(t-1) - 1\}$$

$$\begin{aligned} \hat{h}(t) &:= \min\{h \in \mathbb{Z}_+ : h \geq h(t-1); \\ &\quad M_h(\chi_\varepsilon(t)) \leq 1 - \mu\} \end{aligned}$$

where $t \in \mathbb{Z}_+$; $\mu \in (0, 1)$ is the hysteresis constant; $h(0) = \hat{h}(0)$ with $h(-1) = \underline{h}$; and $M_h(\chi_\varepsilon)$ as in (17).

Remark 2: The supervisory logic (23) is responsible for choosing the horizon $h(t)$ given the state $\chi_\varepsilon(t)$. In (23), \underline{h} , $\underline{h} \geq n$, denotes the minimum horizon whose choice is up to the designer (roughly, the larger \underline{h} , the narrower the frequency bandwidth of the closed-loop system with transition matrix $\mathcal{A} + \mathcal{B}\mathcal{F}_{\underline{h}}$). Stability of the switched system is ensured by the crucial condition $h(t) \geq h(t-1)$ (the admissibility condition, as indicated [21]). In words, the horizon is not allowed to decrease more than one unit at a single time-step, while arbitrary increases of the horizon do not destroy stability. The hysteresis constant μ makes h “easier” to decrease than to increase or stay constant.

4.1. Noiseless Case

Theorem 1: Consider the reachable ANCBI system (1) under the input-increment saturations (6). Let the control increment be given by $\delta u(t) = F_{h(t)}\chi_\varepsilon(t)$ with $F_{h(t)}$ as in (14), $\chi_\varepsilon(t)$ as in (20) and $h(t)$ chosen according to the hysteresis switching logic (23) with $\mu \in [0, 1)$. Then, in the absence of time varying disturbances, viz. $\delta\xi(t) \equiv 0$, $\delta\zeta(t) \equiv 0$, the resulting closed-loop hysteresis switched system yields offset-free tracking of constant set-point r , global asymptotic stability and semi-global exponential stability, irrespective of both the initial state $x(0) \in \mathbb{R}^{n_x}$ and the magnitude of r and of the constant disturbances.

Proof: The proof relies on the fact that, in the presence of constant disturbances, the design is carried out by using the incremental model (20), which turns out to be unaffected by such disturbances. Then, the result directly follows from [21]. \square

4.2. The Horizon Resetting Mechanism in the Noisy Case

As already pointed out, system (20) has not the structure of a control-saturated LTI system for which it makes sense to consider stability and boundedness under arbitrary l_∞ -disturbances. However, one here has to take into account that not only the positional and incremental disturbances are bounded, but also any partial sum of the incremental ones, viz. $\forall t, v \in [0, \infty)$

$$\begin{cases} |\xi(t)| \leq \Xi \Rightarrow \left| \sum_{i=0}^v \delta\xi(t) \right| \leq 2\Xi \\ |n(t)| \leq \bar{\mathcal{N}} \Rightarrow \left| \sum_{i=0}^v \delta n(t) \right| \leq 2\bar{\mathcal{N}} \end{cases} \quad (24)$$

These properties allow one prove the conjecture that, for $h(t)$ sufficiently large, there exists an interval, comprising L consecutive steps, such that the contribution to δu of any sequence $\{\delta n(t)\}_{t=0}^L$, $L \in \bar{\mathcal{L}}$, which enters the integrator modes, might be of the same order of $\sum_{i=0}^L \delta n(t)$. Specifically, the mentioned conjecture relies on the following argument. Let $p := [\delta q' \varepsilon'^-]'$ denote the neutrally stable substate of χ_ε , and assume that the control horizon $h(\cdot)$ grows unbounded. This implies that $\chi_\varepsilon(\cdot)$ is unbounded. As Φ_s is a stability matrix, and $\delta u(t)$ and $\delta\xi(t)$ are bounded, there must be a time t large enough at which $\|\chi_\varepsilon(t)\|^2 = \|\delta s(t)\|^2 + \|p(t)\|^2 \simeq \|p(t)\|^2$. As $\sum_{j=0}^t \delta n(j)$ is bounded, h is chosen after such a large t , according to the restricted system with state $p(t+l) = \hat{p}(t+l) + \tilde{p}(t+l) \cong \hat{p}(t+l)$, where $\hat{p}(t+l)$ is related to the noiseless system while $\tilde{p}(t+l)$ is the response to the bounded term $\sum_{j=t}^{t+l-1} \delta n(j)$. Under these circumstances, at times $t+l$ subsequent to such a large t , $h(t+l) = h(t) - l$ until $\|\hat{p}(t+l)\|$ decreases so as to make $\|\delta s(t+l)\|$ and/or $\|\tilde{p}(t+l)\|$ comparable with $\|\hat{p}(t+l)\|$ and, hence, significant again for the selection of the horizon. This means that a “horizon resetting mechanism” is inherently enforced. Then the conjecture, that will be proved to hold in the remaining part of this paper, is that such a mechanism prevents $h(t)$ (and the plant state) from growing unbounded.

In order to prove the mentioned horizon resetting property, it is convenient to introduce the following lemma which is fundamental for the subsequent developments.

Lemma 1: Let $\chi := \iota + \vartheta$, $\iota := [0'_{n_x} \varepsilon']'$, $\vartheta := [\delta x' 0'_\varepsilon]'$, with \mathcal{F}_h as in (11). Then, $\forall j \geq 1$, one has

$$\mathcal{F}_{h+j}(k+j)\iota = \mathcal{F}_h(k)\iota + \sum_{i=1}^j \mathcal{S}_i \iota \quad (25)$$

$$\mathcal{S}_i := \mathcal{F}_{h+i-1}(k+i-1)\mathcal{B}\mathcal{F}_{h+i}(0), \quad \forall i \in [1, j] \quad (26)$$

Proof: See the Appendix. \square

Remark 3: As can be seen, the proof of the Lemma 1 hinges upon the key property that $\mathcal{A}\iota = \iota$. This property clarifies, a posteriori, that arbitrary time-varying disturbances in the UPS system (1) cannot be directly

injected into the critically unstable modes. Indeed, let $c(t)$ denote such a disturbance and let $w(t) := [0_s' c'(t) 0_\varepsilon']'$. Under these circumstances $\mathcal{A}w = [0'(\Phi_q w)'(H_q w)']'$, and hence, the property given in Lemma 1 would not hold true. In the presence of such disturbances, to the best of the authors' knowledge, one can only avail of the pertinent result given in [21] (viz. the condition $\|\delta c(t)\| < \mu(1 - \mu)/\bar{M}$, where μ is the hysteresis constant in (23) and \bar{M} is as in (18)).

The result of next Theorem 2 hinges upon the following lemma, proved in [21], which the reader is referred to.

Lemma 2 [21]: Consider the model (20) and its related $M_h(\chi_\varepsilon)$ defined in (17). Then, there exist large enough integers h, N , $h - N > 0$, such that the two following inequalities jointly hold

$$M_h(\chi_\varepsilon) \leq \gamma M_{h-N}(\chi_\varepsilon) \quad (27)$$

$$\lambda^h \leq c/N \quad (28)$$

for $\gamma, c > 0$, $\lambda \in (0, 1)$ and $\forall \chi_\varepsilon \in \mathbb{R}^n$. \square

Before proceeding any further, it is worth pointing out that if the switching supervisory mechanism prevents the state χ_ε of IM_ε (20) from becoming unbounded, the same property also holds, as next Lemma 3 proves, for the state x of UPS (1).

Lemma 3: Consider the reachable ANCBI system (1) under the input-increment saturations (6). Let the control increment be given by $\delta u(t) = F_{h(t)} \chi_\varepsilon(t)$ with $F_{h(t)}$ as in (14) and $h(t)$ chosen according to the hysteresis switching logic (23). Then, $\forall t \in \mathbb{Z}_+$, $h(t)$ bounded implies $u(t)$ and $x(t)$ in the UPS (1) bounded.

Proof: See the Appendix. \square

By virtue of Lemma 3, suppose that $\chi_\varepsilon(\cdot)$ (and hence $h(\cdot)$) is bounded. Because $\chi_\varepsilon(\cdot)$ is bounded, $u(\cdot)$ and $x(\cdot)$ are bounded as well. This property allows one to focus the attention only on the state χ_ε .

Theorem 2: Consider the reachable ANCBI system (1) under the input-increment saturations (6). Let the control increment be given by $\delta u(t) = F_{h(t)} \chi_\varepsilon(t)$ with $F_{h(t)}$ as in (14), $\chi_\varepsilon(t)$ as in (20) and $h(t)$ chosen according to the hysteresis switching logic (23). Then, the resulting closed-loop hysteresis switched system is bounded-noise bounded-state l_∞ -stable irrespective of both the initial state $x(0) \in \mathbb{R}^{n_x}$ and the magnitude of $\xi(\cdot)$, $\zeta(\cdot)$ and $r(\cdot)$ ².

²The closed loop system yields also offset-free tracking for the class of disturbances and the reference sequences which become constant in finite time.

Proof: See the Appendix. \square

Theorem 2, along with Theorem 1, completes the extension of the approach of [21] to the present case of set-point tracking under input-increment saturations and persistent disturbances.

Remark 4: Memory/Computational savings. As indicated in [21] and [22] there are some properties of the feedback-gains (11) which can be conveniently exploited for keeping the memory/computational load requirements of the horizon-switching predictive algorithm of this paper at a moderate level. As for memory, it can be shown that the virtual feedback-gains for a given horizon are computable from the ones for a larger horizon. As for computations, in order to perform the admissibility test, it suffices to store the first $n - \rho$ feedback-gains, where ρ denotes the number of the zero roots of the characteristic polynomial of \mathcal{A} , while all the remaining ones in the sequence can be generated recursively.

Remark 5: Partial state information. Suppose that the true plant state $x(t)$ in (1) is replaced by the vector $x(t) + [\vartheta'(t) 0']'$, where $\vartheta(t)$ is a bounded sensor-noise acting on the stable modes of the system. It is immediate to see that the conclusions of Theorem 2 hold true. In such a case the hysteresis switching logic (23) is based on the partial state information $\hat{\chi}_\varepsilon(t) = [\hat{\delta} s'(t) \delta q'(t) \times \varepsilon'(t - 1)]'$, with $\hat{\chi}_\varepsilon(t)$ as in (20), and where $\hat{\delta} s(t)$ is a filtered-estimate of $\delta s(t)$ based on observations $\phi(t) = [\gamma(t) \delta q'(t) \varepsilon'(t - 1)]'$, $\gamma(t) = E \delta s(t) + \varphi(t) \in \mathbb{R}^p$ with $\varphi(\cdot)$ a bounded noise.

5. An Example

Consider the control of the roll angle of an aircraft [30]. The discrete-time system (zero-order hold and sampling time 5 ms) is as follows

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0.9956 & 0.0177 & 0.0004 \\ 0 & 0.7788 & 0.0354 \\ 0 & 0 & 1 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 0.0000 \\ 0.0007 \\ 0.0395 \end{bmatrix} u(t) + \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ c(t) \end{bmatrix} \\ y(t) &= [-3.9528 \quad 0 \quad 0] x(t) + \zeta(t) \end{aligned} \quad (29)$$

System (29) is used to show that, in the presence of input-increment saturations, a non large enough constant horizon \bar{h} makes unstable the closed-loop, even in the absence of disturbances. The control in Fig. 1a is generated by $F_{\bar{h}} \chi(t)$, $\bar{h} = 10$, when the reference is a square wave between -20 and 20 , and

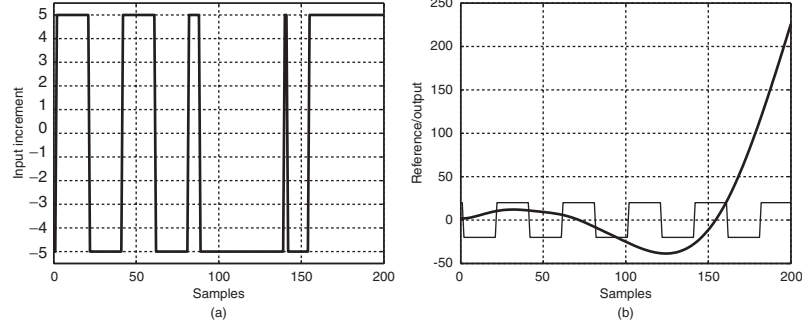


Fig. 1. (a) Saturated input δu ; (b) Divergence trend of output y with $\bar{h} = 10$.

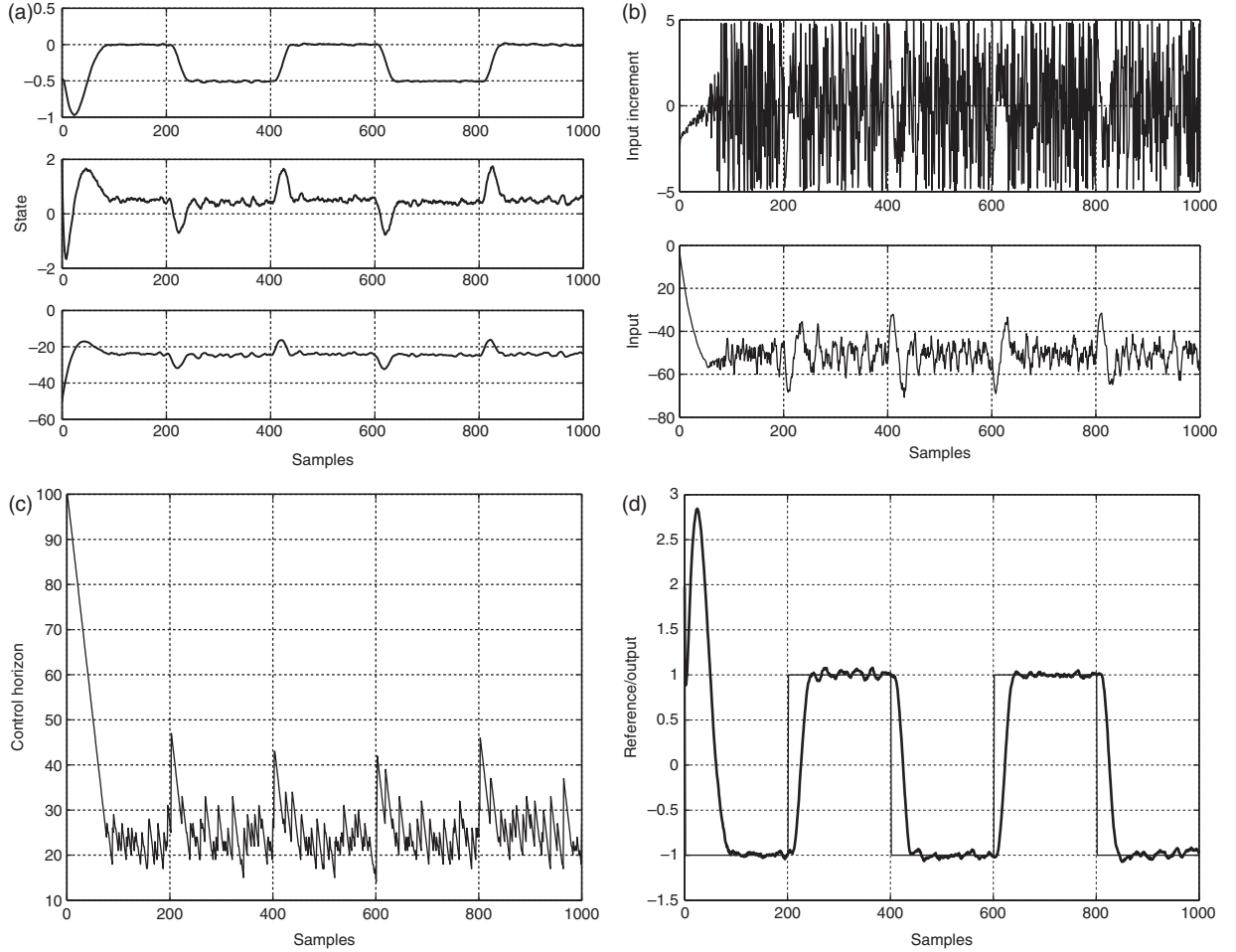


Fig. 2. (a) State x ; (b) Incremental input $\delta u \in [-5, 5]$ and input u ; (c) Horizon h ; (d) Square wave reference r and output y .

the input increment to the plant saturates outside $[-5, 5]$. Fig. 1b shows the related divergence trend of y . We next verify the horizon resetting property proved in Theorem 2. It can be shown that (29) is algebraically equivalent to systems as in (1) with state $x = [x_1 x_2 x_3]'$, where x_1 and x_2 are related to the stable

modes. The aim is to stabilize (29) and make y to track a square wave between $[-1, 1]$ using a control action $\delta u = F_{h(t)} \chi(t)$, which saturates outside $[-5, 5]$. The simulations in Fig. 2 refer to disturbances uniformly distributed, $\xi_1 \in [0 \pm 0.003]$, $\xi_2 \in [1 \pm 0.05]$, $c \in [2 \pm 10^{-5}]$ and $\zeta \in [-1 \pm 0.01]$. As already noted, the

disturbance c , entering the neutrally stable mode x_3 , cannot be a time-varying arbitrary bounded sequence. In order to enforce the horizon resetting mechanism, an initial state $x(0)$ is used so that $\|s(0)\|^2 = x_1^2(0) + x_2^2(0) \ll x_3^2(0) = q^2(0)$. Fig. 2c shows, in agreement with the horizon resetting mechanism, that the control horizon decreases by one at each time-step, irrespective of the disturbances, as long as $\|s(0)\|^2 \ll q^2(0)$. Notice that, depending on the magnitude of the disturbances, the pair $(x(t), u(t))$ remains in a neighborhood of the state-input $([-0.5060, 0.4038, -24.6895], -50.5964)$ and $([0, 0.5145, -23.9977], -50.5964)$ corresponding, in the steady-state with constant disturbances, to $r = 1$, respectively, $r = -1$.

6. Conclusions

This paper provides, relatively alternative approaches yielding comparable feasibility/performance properties, a computationally affordable solution to the set-point tracking problem of discrete-time LTI systems subject to input-increment saturations and in the presence of persistent disturbances of unknown arbitrary magnitude. One of the main contribution of this paper is the proof that all arbitrarily bounded disturbances, that can be in principle tolerated by the positional system under positional input saturations for the pure regulation problem, are also effectively handled by the control algorithm of this paper under input-increment saturations even when the output that is used for tracking is affected by arbitrary bounded disturbances. This is a nontrivial result considering that the disturbances enter the associated incremental model in a way that, seemingly, does not conform with the canonical structure that allows to handle arbitrary bounded disturbances. The proposed solution enjoys the following features: It consists of a supervisory switching control logic whereby a feedback-gain, selected at any time from a family of pre-designed candidate feedback-gains, is switched-on in feedback to the plant according to the information, either complete or partial, on the current plant state; the controller selection is made in accordance with a predictive control philosophy, and each candidate feedback-gain is tuned on to a different horizon in a receding-horizon control sense. It is proved that the adopted switching logic ensures global asymptotic and semi-global exponential stability in the ideal noiseless case, as well as bounded-noise bounded-state l_∞ -stability under the stated persistent arbitrary disturbances.

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Appendix

Proof of Lemma 1: According to Bellman's principle of optimality [16], an optimal trajectory remains such from each intermediate time onward for the cost-to-go. Hence, $\mathcal{F}_h(k+1) = \mathcal{F}_{h-1}(k)\mathcal{A}_h$, $\mathcal{A}_h = \mathcal{A} + \mathcal{B}\mathcal{F}_h(0)$. Consequently, one can write $\forall j \geq 1$

$$\begin{aligned} \mathcal{F}_{h+j}(k+j)\iota &= \mathcal{F}_{h+j-1}(k+j-1)\mathcal{A}_{h+j}\iota = \\ &= \mathcal{F}_{h+j-1}(k+j-1)(\mathcal{A} + \mathcal{B}\mathcal{F}_{h+j}(0))\iota = \\ &= \mathcal{F}_{h+j-1}(k+j-1)\iota + \\ &\quad + \mathcal{F}_{h+j-1}(k+j-1)\mathcal{B}\mathcal{F}_{h+j}(0)\iota \end{aligned}$$

where the last equality holds because, as can be checked, $\mathcal{A}\iota = \iota$. Defining

$$\mathcal{S}_i := \mathcal{F}_{h+i-1}(k+i-1)\mathcal{B}\mathcal{F}_{h+i}(0) \quad (30)$$

by arguing in a similar way, the proof follows by mathematical induction. \square

Proof of Lemma 3: The proof is made by contradiction. Notice first that $h(\cdot)$ is bounded if and only if $\chi_\varepsilon(\cdot)$ is bounded. Rewrite (1) in steady-state for $e(\hat{t}) = 0$, $\hat{t} := h(t) - 1$, to find

$$\begin{bmatrix} I_{n_x} - \Phi & G \\ -H & 0_m \end{bmatrix} \begin{bmatrix} s^s(\hat{t}) \\ q^s(\hat{t}) \\ -u^s(\hat{t}) \end{bmatrix} = \begin{bmatrix} \xi(t-1) \\ c \\ w(t-1) \end{bmatrix} \quad (31)$$

where $w(t-1) := \zeta(t-1) - r(t-1)$. Notice that $x^s(\hat{t}) = [s^s(\hat{t})' q^s(\hat{t})']'$ and $u^s(\hat{t})$ uniquely exist under (7) and only depend on the system matrices and exogenous inputs $\xi(t-1)$, c , $\zeta(t-1)$ and $r(t-1)$. At every time-step t , such signals define a new bounded steady pair $(x^s(\hat{t}), u^s(\hat{t}))$. Suppose now that $u(\cdot)$ grows unbounded. Then, there exists a subsequence $\{t_j\}_{j=1}^\infty$, with $\{h(t_j)\}_{j=1}^\infty$ and bounded initial conditions $(x(t_1), u(t_1))$, such that $\lim_{j \rightarrow \infty} u(t_j) = \infty$. By virtue of (16), the latter condition can be equivalently written as $\lim_{j \rightarrow \infty} \sum_{i=1}^j \delta u_{h(t_i)}(0|\chi_\varepsilon(t_i)) = \infty$. Notice that (15) yields $u_{h(t_j)}(h(t_j) - 1|\chi(t_j)) = u^s(\hat{t}_j)$. Then, $\forall j \in [1, \infty)$, one has

$$\begin{aligned} u^s(\hat{t}_j) &= u(t_j - 1) + \sum_{k=0}^{h(t_j)-1} \delta u_{h(t_j)}(k|\chi_\varepsilon(t_j)) = \\ &= u(t_1) + \sum_{i=1}^{j-1} \delta u_{h(t_i)}(0|\chi_\varepsilon(t_i)) \\ &\quad + \sum_{k=0}^{h(t_j)-1} \delta u_{h(t_j)}(k|\chi_\varepsilon(t_j)) \end{aligned} \quad (32)$$

Because both $u^s(\hat{t}_j)$ and $u(t_1)$ are bounded, (32) implies $\lim_{j \rightarrow \infty} \sum_{k=0}^{h(t_j)-1} \delta u_{h(t_j)}(k|\chi_\varepsilon(t_j)) = \infty$. Finally, the saturation constraints $|\delta u_{h(t_j)}(k|\chi_\varepsilon(t_j))|_i \leq \bar{\Delta}$, $i \in \bar{m}$, $k+1 \in \bar{h}$ contradict the fact that $\lim_{j \rightarrow \infty} h(t_j) < \bar{h}$. Hence, $u(\cdot)$ is bounded. By the same argument it follows that also $q(\cdot)$ is bounded. Indeed, suppose that $\lim_{j \rightarrow \infty} q(t_j) = \infty$, and let $q^s(\hat{t}_j) := q_{h(t_j)}(h(t_j) - 1|\chi(t_j))$ be the steady-state at time-step t_j with prediction horizon $h(t_j)$. One has

$$\begin{aligned} q^s(\hat{t}_j) &= \Phi_q^{h(t_j)} q(t_j - 1) + \sum_{k=0}^{h(t_j)-1} \Phi_q^{h(t_j)-k-1} c \\ &\quad + \sum_{k=0}^{h(t_j)-1} \Phi_q^{h(t_j)-k-1} G_q u_{h(t_j)}(k|\chi_\varepsilon(t_j)) \end{aligned} \quad (33)$$

Hence, $q(t_j - 1)$ unbounded, with c and $u_{h(t_j)}(k|\chi_\varepsilon(t_j))$ bounded contradicts the fact that $\lim_{j \rightarrow \infty} h(t_j) < \bar{h}$. Finally, the boundedness of $s(t)$ directly follows from the fact that $u(t)$ and $\xi(t)$ are bounded and Φ_s is a stability matrix. \square

Proof of Theorem 2: For the reader's convenience the proof of Theorem 2 is divided into three steps illustrating the conceptual flow of the proof.

Step 1 Analysis of the control law.

For the sake of notational simplicity, it will be assumed w.l.o.g. that $\bar{\Delta} = \underline{\Delta}_i = \bar{\Delta}_i$, $\forall i \in \bar{m}$. It is also convenient to define

$$|\delta u_h(\chi_\varepsilon)| \leq b$$

where $|\delta u_h(\chi_\varepsilon)| \leq b$ stands for $|\delta u_h(k|\chi_\varepsilon)|_i \leq b$, $\forall i \in \overline{m}$ e $k+1 \in \overline{h}$, where $|\delta u|_i$ denotes the absolute value of the i^{th} component of δu . Let $\eta := \mu\overline{\Delta}$, with μ as in (23) and where $\overline{\Delta}$ follows from (6). Finally, the following shorthand notation will be used:

$$\delta\hat{\xi}(t) := [\delta\xi'(t)0'_q0_{m'}]', \delta\hat{n}(t) := [0'_s0'_q\delta n'(t)]' \quad (34)$$

Notice that the latter equation, along with (22), implies

$$\delta u_h(\delta\hat{\xi}(t) + \delta\hat{n}(t)) = \delta u_h^s(\delta\hat{\xi}(t)) + \delta u_h^\varepsilon(\delta\hat{n}(t)) \quad (35)$$

Moreover, according to Bellman's principle of optimality, if $\mathcal{A}_h = \mathcal{A} + \mathcal{BF}_h(0)$ it follows that $\delta u_{h-1}(k|\mathcal{A}_h\chi_\varepsilon) = \delta u_h(k+1|\chi_\varepsilon)$. Consequently, if horizon decreases, one can write, $\forall i \in [0, \infty)$

$$\begin{aligned} \delta u_{h(t+i)}(k|\chi_\varepsilon(t+i)) &= \\ &= \delta u_{h(t+i)+i}(k+i|\chi_\varepsilon(t)) + \delta u_\xi^s(t+i) \\ &\quad + \delta u_n^\varepsilon(t+i) \end{aligned} \quad (36)$$

where

$$\begin{aligned} \delta u_\xi^s(t+i) &:= \delta u_{h(t+i)+i-1}^s(k+i-1|\delta\hat{\xi}(t)) \\ &\quad + \dots + \delta u_{h(t+i)}^s(k|\delta\hat{\xi}(t+i-1)) \end{aligned} \quad (37)$$

denotes the contribution of the disturbance which enters the stable modes of χ_ε , while

$$\begin{aligned} \delta u_n^\varepsilon(t+i) &:= \delta u_{h(t+i)+i-1}^\varepsilon(k+i-1|\delta\hat{n}(t)) + \\ &\quad + \dots + \delta u_{h(t+i)}^\varepsilon(k|\delta\hat{n}(t+i-1)) \end{aligned} \quad (38)$$

is the contribution related to the critically unstable ones.

Step 2 Horizon resetting mechanism.

The proof proceeds similarly to [21]. Let, by contradiction, $h(\cdot)$ be unbounded. Then, there exists a subsequence $\{t_j\}_{j=1}^\infty$ such that $\lim_{j \rightarrow \infty} h(t_j) = \infty$ and $h(t) \leq h(t_j)$, $t \leq t_j$. By virtue of Lemma 2, there exist large enough regulation horizons h and integers $L < h$, such that the following inequalities jointly hold

$$|\delta u_{h-L}^s(\delta\hat{\xi})| \leq 2\hat{\Delta}\bar{\lambda}_{\Phi_s}^{h-L+1}\bar{\Xi} \leq \eta_1/L \quad (39)$$

$$|\delta u_{h-L}^\varepsilon(\delta\hat{n})| \leq 2\overline{\Delta M}(h-L+1)^{-1}\overline{\mathcal{N}} \leq \eta_2 \quad (40)$$

$$\begin{aligned} \eta_1 + \eta_2 &= \eta, \hat{\Delta} \in (0, \infty), l \in \overleftarrow{L} \text{ and} \\ M_h(\chi_\varepsilon) &\leq \gamma M_{h-N}(\chi_\varepsilon) \end{aligned} \quad (41)$$

for $\gamma = 1 - 2\eta_2/\overline{\Delta} - [(L+1)/L](\eta_1/\overline{\Delta})$, $N \geq L+1$ and $\forall \chi_\varepsilon \in \mathbb{R}^n$. Notice, in the definition of γ , that $2\eta_2$ is the L -counterpart of $\eta_1(L+1)/L$.

Choose an j so large that, with $\tau := t_j$, $h = h(\tau)$ satisfy (39), (40) and (41). Because of switching criterion (23),

$$|\delta u_{h(\tau)}(\chi_\varepsilon(\tau))| \leq \overline{\Delta} - \eta \quad (42)$$

as $h(\tau-1) \leq h(\tau)$. So, according to Bellman's principle, if $\chi_\varepsilon(\tau+l) = \mathcal{A}_{h(\tau)-l+1}\chi_\varepsilon(\tau+l-1) + [\delta\xi'(\tau+l-1)0'_q\delta n'(\tau+l-1)]'$, $l \in \overleftarrow{L}$, one has

$$|\delta u_{h(\tau)-l}(\chi_\varepsilon(\tau+l))| \leq \overline{\Delta} - [(L-l)/L]\eta_1 \quad (43)$$

provided that, with $h(\tau+l) = h(\tau) - l$, we can write for (38)

$$|\delta u_n^\varepsilon(\tau+l)| \leq \eta_2 \quad (44)$$

Put in other words this means that, for $l \in \overleftarrow{L}$, the control horizon h is allowed to decrease by one unit at each time-step, provided that (44) holds true.

Step 2.1 Horizon $h(t_j)$ is never exceeded.

Let $N \geq L+1$ be the smallest integer at which $M_{h(\tau)-N+1}(\chi_\varepsilon(\tau+N-1)) \leq 1$ and $M_{h(\tau)-N}(\chi_\varepsilon \times (\tau+N)) > 1$. By (41), $\forall k \in \overleftarrow{h(\tau)}$ and $\forall i \in \overline{m}$ one has

$$\begin{aligned} |\delta u_{h(\tau)}(k|\chi_\varepsilon(\tau+N))|_i &\leq \\ &\leq |\delta u_{h(\tau)}(k|\overline{\chi}_\varepsilon)|_i + \eta_1/L + \eta_2 \leq \\ &\leq \overline{\Delta M}_{h(\tau)}(\overline{\chi}_\varepsilon) + \eta_1/L + \eta_2 \leq \\ &\leq \gamma \overline{\Delta M}_{h(\tau)-N}(\overline{\chi}_\varepsilon) + \eta_1/L + \eta_2 \leq \\ &\leq \gamma \overline{\Delta} + \eta_1/L + \eta_2 = \overline{\Delta} - \eta \end{aligned} \quad (45)$$

with $\overline{\chi}_\varepsilon := \mathcal{A}_{h(\tau)-N+1}\chi_\varepsilon(\tau+N-1)$; the first inequality follows from (39) and (40), while the last one follows from $|\delta u_{h(\tau)-N}(k|\overline{\chi}_\varepsilon)|_i = |\delta u_{h(\tau)-N+1}(k+1|\chi_\varepsilon(\tau+N-1))|_i \leq \overline{\Delta}$, $\forall k \in \overleftarrow{h(\tau)-N}$. Therefore $h(t_j+N) \leq h(t_j)$.

Moreover, as in [21], future regulation horizons will never exceed $h(t_j)$. Let $v := t_j + N$ and consider

$$\begin{aligned} |\delta u_{h(v)-1}(k|\chi_\varepsilon(r+1))|_i &= |\delta u_{h(v)-1}(k|\mathcal{A}_{h(v)}\chi_\varepsilon(v))| \\ &\quad + \delta u_{h(v)-1}^s(k|\delta\hat{\xi}(v)) \\ &\quad + \delta u_{h(v)-1}^\varepsilon(k|\delta\hat{n}(v))|_i. \end{aligned}$$

Recall that $|\delta u_{h(v)-1}(k|\mathcal{A}_{h(v)}\chi_\varepsilon(v))|_i = |\delta u_{h(v)}(k+1|\chi_\varepsilon(v))|_i \leq \bar{\Delta} - \eta$. Thus, $h(v+1) \geq h(v)$ implies $|\delta u_{h(v)-1}^\varepsilon(k|\delta \hat{\xi}(v)) + \delta u_{h(v)-1}^\varepsilon \times (k|\delta \hat{n}(v))|_i > \eta$, for some k and i . The latter, in turn, by (39) and (40), yields $h(v) \leq h(t_j) - L$. Consequently,

$$\begin{aligned} |\delta u_{h(t_j)}(k|\chi_\varepsilon(v+1))|_i &= \\ &= |\delta u_{h(t_j)}(k|\mathcal{A}_{h(v)}\chi_\varepsilon(v)) + \\ &\quad + \delta u_{h(t_j)}^\varepsilon(k|\delta \hat{\xi}(v)) + \delta u_{h(t_j)}^\varepsilon(k|\delta \hat{n}(v))|_i \leq \\ &\leq \gamma \bar{\Delta} M_{h(t_j)-L-1}(\mathcal{A}_{h(v)}\chi_\varepsilon(v)) + \eta_1/L + \eta_2 \leq \\ &\leq \gamma \bar{\Delta} M_{h(v)-1}(\mathcal{A}_{h(v)}\chi_\varepsilon(v)) + \eta_1/L + \eta_2 \leq \\ &\leq \gamma(\bar{\Delta} - \eta) + \eta_1/L + \eta_2 < \bar{\Delta} - \eta \end{aligned}$$

Hence, $h(t_j + N + 1) \leq h(t_j)$. By arguing again in a similar way, one proves by mathematical induction that

$$h(t_j + k) \leq h(t_j), \forall k \in \mathbb{Z}_+ \quad (46)$$

provided that (44) holds.

Step 3 *Fulfillment of (44).*

To see this, rewrite (38) for $h(\tau + l) = h(\tau) - l, l \in \overleftarrow{L}$,

$$\begin{aligned} \delta u_n^\varepsilon(\tau + l) &:= \delta u_{h(\tau)-1}^\varepsilon(k + l - 1|\delta \hat{n}(\tau)) + \\ &\quad + \dots + \delta u_{h(\tau)-l}^\varepsilon(k|\delta \hat{n}(\tau + l - 1)) \end{aligned}$$

By virtue of Lemma 1, the j^{th} component of $\delta u_n^\varepsilon(\tau + l)$ is given by

$$\begin{aligned} \delta u_{h(\tau)-l+j}(\tau + l) &= \mathcal{F}_{h(\tau)-l+j}(k + j|\delta \hat{n}(\tau + l - j - 1)) = \\ &= (\mathcal{S}_0 + \mathcal{S}_1 + \dots + \mathcal{S}_j)\delta \hat{n}(\tau + l - j - 1) \end{aligned}$$

where $\mathcal{S}_0 := \mathcal{F}_{h(\tau)-l}(k)$, $\mathcal{S}_i = \mathcal{F}_{h(\tau)-l+i-1}(k + i - 1) \times \mathcal{BF}_{h(\tau)-l+i}(0)$, $\forall i \in [1, j]$ e $\forall j \in [1, l]$. Consequently (38) becomes

$$\begin{aligned} \delta u_n^\varepsilon(\tau + l) &= \mathcal{S}_0 \sum_{j=0}^{l-1} \delta \hat{n}(\tau + j) + \\ &\quad + \mathcal{S}_1 \sum_{j=0}^{l-2} \delta \hat{n}(\tau + j) + \dots + \mathcal{S}_{l-1} \delta \hat{n}(\tau) \end{aligned} \quad (47)$$

Hence, (24) yields

$$|\delta u_n^\varepsilon(\tau + l)| \leq 2(|\mathcal{S}_0| + |\mathcal{S}_1| + \dots + |\mathcal{S}_{l-1}|)\bar{\mathcal{N}}.$$

Notice that $\forall i \in [1, j]$, $\exists \bar{\mathcal{S}} > 0$ such that

$$\begin{aligned} |\mathcal{S}_0| &\leq \bar{\mathcal{S}} |O_{h(\tau)-l}| \\ |\mathcal{S}_i| &\leq \bar{\mathcal{S}} |O_{h(\tau)-l}| |O_{h(\tau)-l+1}| \end{aligned} \quad (48)$$

where O_h stands for a quantity at least of the same order of h^{-1} as $h \rightarrow \infty$ [14]. Finally, because $l \in \overleftarrow{L}$, and $L \rightarrow \mu h$ as $h \rightarrow \infty$ (see [21], proof of Lemma 3, for technical details), one finds

$$\begin{aligned} |\delta u_n^\varepsilon(\tau + l)| &\leq 2 \bar{\mathcal{S}} |O_{h(\tau)-L}| \bar{\mathcal{N}} + \\ &\quad + 2(L-1) \bar{\mathcal{S}} |O_{h(\tau)-L}| |O_{h(\tau)-L+1}| \bar{\mathcal{N}} = \\ &= 2 \bar{\mathcal{S}} (|O_{(1-\mu)h(\tau)}| + \frac{\mu}{1-\mu} |O_{(1-\mu)h(\tau)}|) \bar{\mathcal{N}} = \\ &= 2 \bar{\mathcal{S}} \frac{1}{1-\mu} \|O_{(1-\mu)h(\tau)}\| \bar{\mathcal{N}} \end{aligned} \quad (49)$$

Hence (44) holds and this completes the proof. \square